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Proof of existence of global solutions for *m*-component reaction–diffusion systems with mixed boundary conditions via the Lyapunov functional method

Salem Abdelmalek[1] **and Said Kouachi**[2]

[1] Department of Mathematics and Informatiques, University Centre of Tebessa, 12002 Tebessa, Algeria
[2] University Centre of Khenchela, 40100 Khenchela, Algeria

E-mail: a.salem@gawab.com and kouachi.said@caramail.com

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Abstract
To prove global existence for solutions of *m*-component reaction–diffusion systems presents fundamental difficulties in the case in which some components of the system satisfy Neumann boundary conditions while others satisfy nonhomogeneous Dirichlet boundary conditions and nonhomogeneous Robin boundary conditions. The purpose of this paper is to prove the existence of a global solution using a single inequality for the polynomial growth condition of the reaction terms. Our technique is based on the construction of polynomial functionals. This result generalizes those obtained recently by Kouachi *et al* (at press), Kouachi (2002 *Electron. J. Diff. Eqns* **2002** 1), Kouachi (2001 *Electron. J. Diff. Eqns* **2001** 1) and independently by Malham and Xin (1998 *Commun. Math. Phys.* **193** 287).

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1. Introduction

Recently, global existence for solutions of nonlinear parabolic systems of partial differential equations has been the object of a great deal of research. One of the main results of these studies was obtained by Morgan [5], where all the components satisfy the same boundary conditions (Neumann or Dirichlet), and where again the reaction terms are polynomially bounded and satisfy a set of *m* inequalities. In 1993, Hollis [6] completed the work of Morgan and established the global existence in the presence of mixed boundary conditions if certain structure requirements are placed on the system. The results obtained in this work represent the proof of global existence of solutions with Neumann, Dirichlet, nonhomogeneous Robin and a mixture of Dirichlet with nonhomogeneous Robin conditions, and where again the reaction terms are polynomially growth but satisfy a single inequality. The importance of these results is that many systems satisfy our conditions and Morgan and Hollis's ones. Moreover, there are some systems that satisfy our conditions but not theirs (Morgan and Hollis); see, for instance, the last example of Kouachi's article [2].

All along the paper, we will use the following notations and assumptions: we denote by $m \geqslant 2$ the number of equations of the system (i.e. m-component), and for all $i = 1, \ldots, m$:

$$\frac{\partial u_i}{\partial t} - a_i \Delta u_i = f_i(U) \text{ in } \Omega \times \{t > 0\} \tag{1.1}$$

with the boundary conditions

$$\lambda_i u_i + (1 - \lambda_i)\partial_\eta u_i = \beta_i \qquad \text{on} \quad \partial\Omega \times \{t > 0\} \tag{1.2}$$

and the initial data

$$u_i(0, x) = u_i^0(x) \qquad \text{on} \quad \Omega. \tag{1.3}$$

(i) For nonhomogeneous Robin boundary conditions, we use

$$0 < \lambda_i < 1, \qquad \beta_i \geqslant 0, \qquad i = 1, \ldots, m.$$

(ii) For homogeneous Neumann boundary conditions, we use

$$\lambda_i = \beta_i = 0, \qquad i = 1, \ldots, m.$$

(iii) For homogeneous Dirichlet boundary conditions, we use

$$1 - \lambda_i = \beta_i = 0, \qquad i = 1, \ldots, m.$$

(iv) For a mixture of homogeneous Dirichlet with nonhomogeneous Robin boundary conditions, we use $\exists i = 1, \ldots, m : 1 - \lambda_i = \beta_i = 0$ and $0 < \lambda_j < 1, \beta_j \geqslant 0, j = 1, \ldots, m$ with $i \neq j$,

where $U = (u_i)_{i=1}^m$ and a_i are positive constants for all $i = 1, \ldots, m$; $i = 1, \ldots, m : 0 \leqslant \lambda_i \leqslant 1$ and $\beta_i \geqslant 0$ are in $C^1(\partial\Omega \times \mathbb{R}_+)$.

The initial data are assumed to be non-negative.

(A1) The functions f_i are continuously differentiable on $\mathbb{R}_+^m$ for all $i = 1, \ldots, m$, satisfying $f_i(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) \geqslant 0$, for all $u_i \geqslant 0$; $i = 1, \ldots, m$.

(A2) We suppose that the functions f_i are of polynomial growth (see Hollis and Morgan [7]). This means that for all $i = 1, \ldots, m$, there exists an integer $N \geqslant 1$ such that

$$|f_i(U)| \leqslant C_1 \left(1 + \sum_{i=1}^{m} u_i\right)^N \qquad \text{on} \quad (0, +\infty)^m, \tag{1.4}$$

(A3) and satisfy

$$\sum_{i=1}^{m-1} D_i f_i(U) + f_m(U) \leqslant C_2 \left(1 + \sum_{i=1}^{m} u_i\right) \tag{1.5}$$

for all $u_i \geqslant 0, i = 1, \ldots, m$, and all constants $D_i \geqslant \overline{D_i}, i = 1, \ldots, m$, where $\overline{D_i}, i = 1, \ldots, m$, are sufficiently large positive constants, and C_1 and C_2 are the positive and uniformly bounded functions defined on $\mathbb{R}_+^m$.

Put $A_{ij} = \frac{a_i+a_j}{2\sqrt{a_i a_j}}$ for all $i, j = 1, \ldots, m$. Let $\theta_i, i = 1, \ldots, m-1$, be positive constants such that

$$K_l^l > 0; \qquad l = 2, \ldots, m, \tag{1.6}$$

where

$$K_l^r = K_{r-1}^{r-1} \cdot K_l^{r-1} - \left[H_l^{r-1}\right]^2, \qquad r = 3, \ldots, l,$$

$$H_l^r = \det_{1 \leqslant i,j \leqslant l}\left((a_{i,j})_{\substack{i \neq l,\ldots,r+1 \\ j \neq l-1,\ldots,r}}\right) \cdot \prod_{k=1}^{k=r-2} (\det k)^{2^{(r-k-2)}}, \qquad r = 3, \ldots, l-1,$$

$$K_l^2 = \underbrace{a_1 a_l \prod_{k=1}^{l-1} \theta_k^{2(p_k+1)^2} \cdot \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positivevalue}} \cdot \left[\prod_{k=1}^{l-1} \theta_k^2 - A_{1l}^2\right]$$

and

$$H_l^2 = \underbrace{a_1\sqrt{a_2 a_l}\,\theta_1^{2(p_1+1)^2} \prod_{k=2}^{l-1} \theta_k^{(p_k+2)^2+(p_k+1)^2} \cdot \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positivevalue}} \cdot \left[\theta_1^2 A_{2l} - A_{12}A_{1l}\right].$$

Here, $\det_{1 \leqslant i,j \leqslant l}\left((a_{i,j})_{\substack{i \neq l,\ldots,r+1 \\ j \neq l-1,\ldots,r}}\right)$ denotes the determinant of r square symmetric matrix obtained from $(a_{i,j})_{1 \leqslant i,j \leqslant m}$ by removing the $(r+1)$th, $(r+2)$th, $\ldots$, lth rows and the rth, $(r+1)$th, $\ldots$, $(l-1)$th columns. The elements of the matrix are

$$a_{ij} = \frac{a_i + a_j}{2} \theta_1^{p_1^2} \cdots \theta_{(i-1)}^{p_{(i-1)}^2} \theta_i^{(p_i+1)^2} \cdots \theta_{j-1}^{(p_{(j-1)}+1)^2} \theta_j^{(p_j+2)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+2)^2}. \tag{1.7}$$

The main result of this paper, to be proved in section 4, reads as follows:

Theorem 1. *Suppose that the functions $f_i, i = 1, \ldots, m$, are of polynomial growth and satisfy condition (1.5) for some positive constants $D_i, i = 1, \ldots, m$, sufficiently large. Let $(u_1(t,.), u_2(t,.), \ldots, u_m(t,.))$ be a solution of (1.1)–(1.3) and let*

$$L(t) = \int_\Omega H_{p_m}(u_1(t,x), u_2(t,x), \ldots, u_m(t,x))\,\mathrm{d}x, \tag{1.8}$$

where

$$H_{p_m}(u_1, \ldots, u_m) = \sum_{p_{m-1}=0}^{p_m} \cdots \sum_{p_1=0}^{p_2} C_{p_m}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2} u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{p_m-p_{m-1}},$$

with p_m being a positive integer and $C_{p_j}^{p_i} = \frac{p_j!}{p_i!(p_j-p_i)!}$.

Then the functional L is uniformly bounded on the interval $[0, T^*]$, $T^* < T_{\max}$.

Corollary 1. *Under the assumptions of theorem 1, all solutions of (1.1)–(1.3) with positive initial data in $L^\infty(\Omega)$ are in $L^\infty(0, T^*; L^p(\Omega))$ for some $p \geqslant 1$.*

Proposition 1. *Under the assumptions of theorem 1 and that condition (1.4) is satisfied, all solutions of (1.1)–(1.3) with positive initial data in $L^\infty(\Omega)$ are global for some $p > \frac{Nn}{2}$.*

2. Previous results

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted, respectively, by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p \,dx, \tag{2.1a}$$

$$\|u\|_\infty = \max_{x\in\Omega}|u(x)|. \tag{2.1b}$$

In the two-component case, where $f_1(u_1;u_2) = -f_2(u_1;u_2) = -u_1u_2^\beta$, Alikakos [8] established the global existence and L^∞-bounds of solutions when $1 \leqslant \beta < \frac{n+2}{n}$. Masuda [9] showed that the solutions to this system exist globally for every $\beta \geqslant 1$. Haraux and Youkana [10] simplified the demonstration of Masuda [9] by using techniques based on Lyapunov functionals. They could handle nonlinearities $f_1(u_1;u_2) = -f_2(u_1;u_2) = -u_1F(u_2)$ satisfying the condition

$$\lim_{s\to+\infty}\left[\frac{\log(1+F(s))}{s}\right] = 0 \tag{2.2}$$

which means that $F(s)$ is of sub-exponential growth. Kouachi and Youkana [11] generalized the results of Haraux and Youkana [10]; they added $-c\Delta u_1$ to the right-hand side of the second equation of the system with the reaction terms $f_1(u_1;u_2) = -\lambda f(u_1;u_2)$ and $f_2(u_1;u_2) = +\mu f(u_1;u_2)$ requiring the condition

$$\lim_{s\to+\infty}\left[\frac{\log(1+f(r+s))}{s}\right] < \alpha^* \qquad \text{for} \quad r>0,$$

with

$$\alpha^* = \frac{2a_1a_2}{n(a_1-a_2)^2\left\|u_1^0\right\|_\infty}\min\left\{\frac{\lambda}{\mu}, \frac{(a_1-a_2)}{c}\right\},$$

where the positive diffusion coefficients a_1, a_2 satisfy $a_1 > a_2$ and c, λ, μ are positive constants. This condition reflects the weak exponential growth of the reaction term f.

In [12], Hollis, Martin and Pierre established the global existence of positive solutions for the system with the boundary conditions (1.2), $i = 1, 2$, $\beta_1, \beta_2 \geqslant 0$ and $0 < \lambda_1; \lambda_2 < 1$, $\lambda_1 = \lambda_2 = 1$, or $\lambda_1 = \lambda_2 = 0$. Also $\beta_1 = \beta_2 = 0$ if $\lambda_1 = \lambda_2 = 0$ and where again the reaction terms are continuously differentiable functions and satisfy the conditions: for each $r > 0$ there are numbers $L_0(r)$ and $\mu_0(r)$ such that

$$\begin{cases}\gamma \geqslant 1, |f_2(u_1,u_2)| \leqslant L_0(r)(1+u_2)^\gamma \\ f_1(u_1,u_2) + f_2(u_1,u_2) \leqslant \mu_0(r),\end{cases}$$

with $r \leqslant u_2$. ($L_0(r)$ and $\mu_0(r)$ are independent of $t > 0$.)

Moreover, the solution is uniformly bounded in t.

But under the conditions of the reaction term that we use in studying m-component, Kouachi has studied two-component (see Kouachi [3]), and independently by Malham and Xin [4], three-component (see Kouachi [2]), but he could not generalize m-component. After we studied four-component (see Kouachi *et al* [1]), modified Dodgson's algorithm with a proof (see Kouachi *et al* [13]), we could simply study five-component and deduce m-component.

Many authors dealt with the m-component system (see [5–7, 14–19]).

In [5], Morgan generalized the results of Hollis, Martin and Pierre (first applied to two-component reaction–diffusion systems [12]) to establish the global existence for solutions of m-component systems ($m \geqslant 2$) with the boundary conditions (1.2), where

$$0 < \lambda_i < 1 \text{ or } \lambda_i = 1 \quad \text{and} \quad \beta_i \geqslant 0, \qquad i = 1,\ldots,m \tag{2.3}$$

or

$$\lambda_i = \beta_i = 0, \qquad i = 1, \ldots, m \tag{2.4}$$

and where the reaction terms are polynomially bounded and satisfy, in the case of our system, the following conditions:

$$\sum_{j=1}^{k} \alpha_{kj} f_j(U) \leqslant C_3 \left(1 + \sum_{i=1}^{m} u_i\right) \qquad \text{for} \quad U \in \mathbb{R}_+^m, \quad k = 1, \ldots, m, \tag{2.5}$$

where α_{kj} is a positive real, C_3 constant that is independent of U. $|f_i(.,.,U)|$, $i = 1, \ldots, m$, is bounded above by a polynomial in $u_1, u_2, \ldots, u_m$.

Formula (2.5) is a common form of Morgan's 'Intermediate Sums' condition. Although it is simple and arises in many applications and is used technically in an extension of a duality argument, it is a set of m inequalities. But our assumption (1.5) is more applied because it is one inequality only.

Martin and Pierre [20] and Hollis [6] extended the results, under the same conditions, to the boundary conditions (1.2) where in (2.3), they took

$$0 \leqslant \lambda_i \leqslant 1 \text{ or } \lambda_i = 1 \text{ and } \beta_i \geqslant 0, \qquad i = 1, \ldots, m$$

but they imposed conditions of the form (2.5), at the same time, on the reaction terms whose corresponding components of the solution satisfy Neumann boundary conditions and on the others which satisfy Dirichlet boundary conditions. In other terms they imposed to the reaction terms to satisfy a set of m inequalities.

3. Preliminary observations

It is well known that to prove the global existence of solutions to (1.1)–(1.3) (see Henry [21]), it suffices to derive a uniform estimate of $\|f_i(u_1, u_2, \ldots, u_m)\|_p$, $i = 1, \ldots, m$, on $[0; T_{\max}[$ in the space $L^p(\Omega)$ for some $p > n/2$. Our aim is to construct polynomial Lyapunov functionals allowing us to obtain L^p-bounds on u_i that lead to global existence for all $i = 1, \ldots, m$. Since the functions f_i are continuously differentiable on $\mathbb{R}_+^m$ for all $i = 1, \ldots, m$, then for any initial data in $C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$O = -\begin{pmatrix} a_1\Delta & 0 & \cdots & 0 \\ 0 & a_2\Delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m\Delta \end{pmatrix}. \tag{3.1}$$

Under these assumptions, the following local existence result is well known (see Friedman [22] and Pazy [23]).

Remark 1. Assumption (A1) contains smoothness and quasipositivity conditions that guarantee local existence of solutions and non-negativity of solutions as long as they exist, via the maximum principle (see Smoller [24]). Assumption (A3) is the usual polynomial growth condition necessary to obtain uniform bounds from p-dependent L^P estimates. (See Hollis and Morgan [16].)

Proposition 2. *The system (1.1)–(1.3) admits a unique, classical solution* $(u_1; u_2; \ldots, u_m)$ *on*$(0, T_{\max}[\times\Omega$.

$$\text{If } T_{\max} < \infty \text{ then } \lim_{t\nearrow T_{\max}} \sum_{i=1}^{m} \|u_i(t,.)\|_\infty = \infty, \tag{3.2}$$

where $T_{\max}\left(\left\|u_1^0\right\|_\infty, \left\|u_2^0\right\|_\infty, \ldots, \left\|u_m^0\right\|_\infty\right)$ *denotes the eventual blow-up time.*

Remark 2. This proposition seems to be well-known (Henry [21]). Nevertheless, we could not find it in the literature in the form stated here and in the book of Rothe ([25, pp 111–8 with proof]). Usually, the explosion property (3.2) is only stated for some norm involving smoothness, but not the L_∞-norm.

4. Proof of the main result

For the proof of theorem 1, we need some preparatory lemmas, which are proved in the appendix.

Lemma 1. *Let* H_{p_m} *be the homogeneous polynomial defined by (1.8). Then*

$$\partial_{u_1} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{(p_1+1)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+1)^2} \times u_1^{p_1} u_2^{p_2-p_1} u_3^{p_3-p_2} \cdots u_m^{(p_m-1)-p_{m-1}}, \tag{4.1}$$

for all $i = 2, \ldots, m-1$*:*

$$\partial_{u_i} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{i-1}^{p_{(i-1)}^2} \theta_i^{(p_i+1)2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+1)^2} \times u_1^{p_1} u_2^{p_2-p_1} u_3^{p_3-p_2} \cdots u_m^{(p_m-1)-p_{m-1}} \tag{4.2}$$

and

$$\partial_{u_m} H_{p_m} = p_m \sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \cdots C_{p_3}^{p_2} C_{p_2}^{p_1} \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2} \times u_1^{p_1} u_2^{p_2-p_1} u_3^{p_3-p_2} \cdots u_m^{(p_m-1)-p_{m-1}}. \tag{4.3}$$

Lemma 2. *The second partial derivatives of* H_{p_m} *are given by*

$$\partial_{u_1^2} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_2=0}^{p_3} \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \times \theta_1^{(p_1+2)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{(p_m-2)-p_{m-1}}, \tag{4.4}$$

for all $i = 2, \ldots, m-1$ *:*

$$\partial_{u_i^2} H_n = p_m(p_m-1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \times \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{i-1}^{p_{i-1}^2} \theta_i^{(p_i+2)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} \cdot u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{(p_m-2)-p_{m-1}}, \tag{4.5}$$

for all $2 \leqslant i < j \leqslant m$:

$$\partial_{u_i u_j} H_n = p_m(p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \times \theta_1^{p_1^2} \cdots \theta_{i-1}^{p_{i-1}^2} \theta_i^{(p_i+1)^2} \cdots \theta_{j-1}^{(p_{j-1}+1)^2} \theta_j^{(p_j+2)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+2)^2} \times u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{(p_m-2)-p_{m-1}}. \quad (4.6)$$

Finally,

$$\partial_{u_m^2} H_n = p_m(p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2} \cdot u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{(p_m-2)-p_{m-1}}. \quad (4.7)$$

Lemma 3. *Let A be the m square symmetric matrix defined by* $A = (a_{ij})_{1\leqslant i,j\leqslant m}$, *then we get this property*[3]

$$\begin{cases} K_m^m = \det m \cdot \prod_{k=1}^{k=m-2} (\det k)^{2^{(m-K-2)}}, & m > 2 \\ K_2^2 = \det 2, \end{cases} \quad (4.8)$$

where

$$K_m^l = K_{l-1}^{l-1} \cdot K_m^{l-1} - \left(H_m^{l-1}\right)^2, \qquad l = 3, \ldots, m, \quad (4.8a)$$

$$H_m^l = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m,\ldots,l+1 \\ j\neq m-1,\ldots,l}}\right) \cdot \prod_{k=1}^{k=l-2} (\det k)^{2^{(l-K-2)}}, \qquad l = 3, \ldots, m-1, \quad (4.8b)$$

$$K_m^2 = a_{11}a_{mm} - (a_{1m})^2, \quad (4.8c)$$

$$H_m^2 = a_{11}a_{2m} - a_{12}a_{1m}. \quad (4.8d)$$

Proof of theorem 1. Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_\Omega \partial_t H_{p_m}\, \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^m \partial_{u_i} H_{p_m} \frac{\partial u_i}{\partial t}\, \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^m \partial_{u_i} H_{p_m} (a_i \Delta u_i + f_i)\, \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^m a_i \partial_{u_i} H_{p_m} \Delta u_i\, \mathrm{d}x + \int_\Omega \sum_{i=1}^m \partial_{u_i} H_{p_m} f_i\, \mathrm{d}x \\ &= I + J. \end{aligned} \quad (4.9)$$

$$I = \int_\Omega \sum_{i=1}^m a_i \partial_{u_i} H_{p_m} \Delta u_i\, \mathrm{d}x$$

$$J = \int_\Omega \sum_{i=1}^m \partial_{u_i} H_{p_m} f_i\, \mathrm{d}x.$$

[3] This property is in the domain of linear algebra.

Using Green's formula, we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega} \sum_{i=1}^{m} a_i \partial_{u_i} H_{p_m} \partial_\eta u_i \, \mathrm{d}s \tag{4.10}$$

and

$$I_2 = -\int_{\Omega} \left[\left(\left(\frac{a_i + a_j}{2} \partial_{u_j u_i} H_{p_m} \right)_{1\leqslant i,j\leqslant m} \right) T \right] \cdot T \, \mathrm{d}x \tag{4.11}$$

for $p_1 = 0, \ldots, p_2$, $p_2 = 0, \ldots, p_3, \ldots, p_{m-1} = 0, \ldots, p_m - 2$ and $T = (\nabla u_1, \nabla u_2, \ldots, \nabla u_m)^t$. □

Applying lemmas 1 and 2, we get

$$\left(\frac{a_i + a_j}{2} \partial_{u_j u_i} H_{p_m} \right)_{1\leqslant i,j\leqslant m} = p_m(p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \left((a_{ij})_{1\leqslant i,j\leqslant m} \right) u_1^{p_1} \cdots u_m^{(p_m-2)-p_{m-1}} \tag{4.12}$$

when $(a_{ij})_{1\leqslant i,j\leqslant m}$ is a matrix defined in formula (1.7).

We prove that there exists a positive constant C_4 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \leqslant C_4 \text{ for all } t \in [0, T_{\max}[\tag{4.13}$$

and that

$$I_2 \leqslant 0 \tag{4.14}$$

for several boundary conditions.

(i) If $i = 1, \ldots, m$: $0 < \lambda_i < 1$, using the boundary conditions (1.2) we get

$$I_1 = \int_{\partial\Omega} \sum_{i=1}^{m} a_i \partial_{u_i} H_{p_m} (\gamma_i - \alpha_i u_i) \, \mathrm{d}s,$$

where $\alpha_i = \frac{\lambda_i}{1-\lambda_i}$ and $\gamma_i = \frac{\beta_i}{1-\lambda_i}$, $i = 1, \ldots, m$. Since $H(U) = \sum_{i=1}^{m} a_i \partial_{u_i} H_{p_m} (\gamma_i - \alpha_i u_i) = P_{n-1}(U) - Q_n(U)$, where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n and since the solution is positive, then

$$\limsup_{\sum_{i=1}^{m} |u_i| \to +\infty} H(U) = -\infty \tag{4.15}$$

which prove that H is uniformly bounded on $\mathbb{R}_+^m$ and consequently (4.13).

(ii) If $\forall i = 1, \ldots, m : \lambda_i = 0$, then $I_1 = 0$ on $[0, T_{\max}[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{\max}[\times\Omega$ implies $\partial_\eta u_i \leqslant 0, \forall i = 1, \ldots, m$, on $[0, T_{\max}[\times\partial\Omega$. Consequently, one gets again (4.13) with $C_4 = 0$.

(iv) If one or two or three … $(m-1)$ of the components of the solution satisfy homogeneous Dirichlet boundary conditions and the other (others) satisfies the nonhomogeneous Robin conditions, for example, $u_1 = 0$, $\lambda_i u_i + (1-\lambda_i)\partial_\eta u_i = \beta_i$, $i = 2, \ldots, m$ on $[0, T_{\max}[\times\partial\Omega$ with $0 < \lambda_i < 1$, $\beta_i \geqslant 0$, $i = 2, \ldots, m$, then following the same reasoning as above we get

$$\limsup_{\sum_{i=2}^{m} |u_i| \to +\infty} H(0, u_2, \ldots, u_m) = -\infty \tag{4.16}$$

and then (4.13).

Now we prove (4.14). $(a_{ij})_{1\leqslant i,j\leqslant m}$ is a matrix defined in formula (1.7).

The quadratic forms (with respect to ∇u_i, $i = 1, \ldots, m$) associated with the matrices $(a_{ij})_{1\leqslant i,j\leqslant m}$, $p_1 = 0, \ldots, p_2$, $p_2 = 0, \ldots, p_3, \ldots, p_{m-1} = 0, \ldots, p_m - 2$ are positive since their main determinants $\det 1, \det 2, \ldots \det m$ are also positive. To see this, we have the following:

(*) $\det 1 = a_1\theta_1^{(p_1+2)^2}\theta_2^{(p_2+2)^2}\cdots\theta_{(m-1)}^{(p_{(m-1)}+2)^2} > 0$ for $p_1 = 0, \ldots, p_2, p_2 = 0, \ldots, p_3 \cdots p_{m-1} = 0, \ldots, p_m - 2$.

(**) According to lemma 3, we get

$$\det 2 = K_2^2 = a_1a_2\theta_1^{2(p_1+1)^2}\prod_{k=2}^{m-1}\theta_k^{2(p_k+2)^2}\left[\theta_1^2 - A_{12}^2\right],$$

using (1.6) for $l = 2$ we get $\det 2 > 0$.

(***) Again according to lemma 3, we have

$$K_3^3 = \det 3 \det 1,$$

but $\det 1 > 0$, thus $\operatorname{sign}(K_3^3) = \operatorname{sign}(\det 3)$.

Using (1.6) for $l = 3$ we get $\det 3 > 0$.

(****) We suppose $\det k > 0$ $k = 1, 2, \ldots, l-1$ and prove that $\det l > 0$

$$\det k > 0, k = 1, \ldots, (l-1) \Rightarrow \prod_{k=1}^{k=l-2}(\det k)^{2^{(l-K-2)}} > 0 \tag{4.17}$$

from lemma 3 $K_l^l = \det l \cdot \Pi_{k=1}^{k=l-2}(\det k)^{2^{(l-K-2)}}$, and from (4.17), we get $\operatorname{sign}\left(K_l^l\right) = \operatorname{sign}(\det l)$ but $K_l^l > 0$, from (1.6), thus $\det l > 0$.

We get (4.14).

Now we prove J-bounded (4.9).

Substituting the expressions of the partial derivatives given by lemma 1 into the second integral (4.9) yields

$$J = \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1}\cdots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}\cdots C_{p_2}^{p_1}u_1^{p_1}u_2^{p_2-p_1}\cdots u_m^{p_m-1-p_{m-1}}\right]$$
$$\times\left(\prod_{i=1}^{m-1}\theta_i^{(p_i+1)^2}f_1 + \sum_{j=2}^{m-1}\prod_{k=1}^{j-1}\theta_k^{p_k^2}\prod_{i=j}^{m-1}\theta_i^{(p_i+1)^2}f_j + \prod_{i=1}^{m-1}\theta_i^{p_i^2}f_m\right)\mathrm{d}x$$
$$= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1}\cdots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}\cdots uC_{p_21}^{p_1p_1}u_2^{p_2-p_1}\cdots u_m^{p_m-1-p_{m-1}}\right]$$
$$\times\left(\frac{\Pi_{i=1}^{m-1}\theta_i^{(p_i+1)^2}}{\Pi_{i=1}^{m-1}\theta_i^{p_i^2}}f_1 + \sum_{j=2}^{m-1}\frac{\Pi_{k=1}^{j-1}\theta_k^{p_k^2}\Pi_{i=j}^{m-1}\theta_i^{(p_i+1)^2}}{\Pi_{i=1}^{m-1}\theta_i^{p_i^2}}f_j + f_m\right)\prod_{i=1}^{m-1}\theta_i^{p_i^2}\,\mathrm{d}x$$
$$= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1}\cdots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}\cdots C_{p_2}^{p_1}u_1^{p_1}u_2^{p_2-p_1}\cdots u_m^{p_m-1-p_{m-1}}\right]$$
$$\times\left(\prod_{i=1}^{m-1}\frac{\theta_i^{(p_i+1)^2}}{\theta_i^{p_i^2}}f_1 + \sum_{j=2}^{m-1}\prod_{i=j}^{m-1}\frac{\theta_i^{(p_i+1)^2}}{\theta_i^{p_i^2}}f_j + f_m\right)\prod_{i=1}^{m-1}\theta_i^{p_i^2}\,\mathrm{d}x.$$

Using condition (1.5), we deduce

$$J \leqslant C_5 \int_\Omega \left[\sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \cdots C_{p_m-1}^{p_{m-1}} u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{p_m-1-p_{m-1}} \left(1+\sum_{i=1}^m u_i\right)\right] \mathrm{d}x.$$

To prove that the functional L is uniformly bounded on the interval $[0, T^*]$, we first write

$$\sum_{p_{m-1}=0}^{p_m-1} \cdots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \cdots C_{p_m-1}^{p_{m-1}} u_1^{p_1} u_2^{p_2-p_1} \cdots u_m^{p_m-1-p_{m-1}} \left(1+\sum_{i=1}^m u_i\right) = R_{p_m}(U) + S_{p_m-1}(U),$$

where $R_{p_m}(U)$ and $S_{p_m-1}(U)$ are two homogeneous polynomials of degrees p_m and $p_m - 1$, respectively. First, since the polynomials H_{p_m} and R_{p_m} are all of degree p_m, there exists a positive constant C_6 such that

$$\int_\Omega R_{p_m}(U)\,\mathrm{d}x \leqslant C_6 \int_\Omega H_{p_m}(U)\,\mathrm{d}x, \tag{4.18}$$

then applying Hölder's inequality to the integral $\int_\Omega S_{p_m-1}(U)\,\mathrm{d}x$, one gets

$$\int_\Omega S_{p_m-1}(U)\,\mathrm{d}x \leqslant (\mathrm{meas}\,\Omega)^{\frac{1}{p_m}} \left(\int_\Omega \left(S_{p_m-1}(U)\right)^{\frac{p_m}{p_m-1}} \mathrm{d}x\right)^{\frac{p_m-1}{p_m}}.$$

Since for all $u_1, u_2, \ldots, u_{m-1} \geqslant 0$ and $u_m > 0$,

$$\frac{\left(S_{p_m-1}(U)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(U)} = \frac{\left(S_{p_m-1}(x_1, x_2, \ldots, x_{m-1}, 1)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \ldots, x_{m-1}, 1)},$$

where $\forall i \in \{1, 2, \ldots, m-1\} : x_i = \frac{u_i}{u_{i+1}}$ and

$$\lim_{x_i \to +\infty} \frac{\left(S_{p_m-1}(x_1, x_2, \ldots, x_{m-1}, 1)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(x_1, x_2, \ldots, x_{m-1}, 1)} < +\infty,$$

one asserts that there exists a positive constant C_7 such that

$$\frac{\left(S_{p_m-1}(U)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}(U)} \leqslant C_7, \qquad \text{for all} \quad u_1, u_2, \ldots, u_m \geqslant 0. \tag{4.19}$$

Hence, the functional L satisfies the differential inequality

$$L'(t) \leqslant C_8 L(t) + C_9 L^{\frac{p_m-1}{p_m}}(t),$$

which for $Z = L^{\frac{1}{p_m}}$ can be written as

$$p_m Z' \leqslant C_8 Z + C_9. \tag{4.20}$$

A simple integration gives the uniform bound of the functional L on the interval $[0, T^*]$; this ends the proof of the theorem. □

Proof of corollary 1. The proof of this corollary is an immediate consequence of theorem 1 and the inequality

$$\int_\Omega \left(\sum_{i=1}^m u_i(t, x)\right)^p \mathrm{d}x \leqslant C_{10} L(t) \qquad \text{on} \quad [0, T^*], \tag{4.21}$$

for some $p \geqslant 1$. □

Proof of proposition 1. From corollary 1, there exists a positive constant C_{11} such that

$$\int_\Omega \left(\sum_{i=1}^m u_i(t, x) + 1\right)^p \mathrm{d}x \leqslant C_{11} \qquad \text{on} \quad [0, T_{\max}[. \tag{4.22}$$

From (1.4) we have

$$\forall i \in \{1, 2, \ldots, m\}: \quad |f_i(U)|^{\frac{p}{N}} \leqslant C_{12}(U)\left(\sum_{i=1}^{m} u_i(t,x)\right)^p \qquad \text{on} \quad [0, T_{\max}[\times\Omega. \tag{4.23}$$

Since $u_1, u_2, \ldots, u_m$ are in $L^\infty(0, T^*; L^p(\Omega))$ and $\frac{p}{N} > \frac{n}{2}$, then from the preliminary observations the solution is global. □

5. Examples

In this section, we will examine two particular examples of biochemical and chemical models. In order to illustrate the applicability of corollary 1 and proposition 1, we assume that all reactions take place in a bounded domain Ω with a smooth boundary $\partial\Omega$.

Example 1. Let us begin with the following reaction:

$$U_1 + U_2 \underset{k_2}{\overset{k_1}{\rightleftarrows}} U_3, \qquad U_1 + U_4 \underset{k_4}{\overset{k_3}{\rightleftarrows}} U_5, \qquad U_2 + U_6 \underset{k_6}{\overset{k_5}{\rightleftarrows}} U_4. \tag{5.1}$$

This leads to the six-component reaction–diffusion system:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - a_1\Delta u_1 &= -k_1u_1u_2 - k_3u_1u_4 + k_2u_3 + k_4u_5,\\
\frac{\partial u_2}{\partial t} - a_2\Delta u_2 &= -k_1u_1u_2 + k_2u_3 - k_5u_2u_6 + k_6u_4,\\
\frac{\partial u_3}{\partial t} - a_3\Delta u_3 &= k_1u_1u_2 - k_2u_3 + k_5u_2u_6 - k_6u_4,\\
\frac{\partial u_4}{\partial t} - a_4\Delta u_4 &= -k_3u_1u_4 + k_4u_5 + k_5u_2u_6 - k_6u_4,\\
\frac{\partial u_5}{\partial t} - a_5\Delta u_5 &= k_3u_1u_4 - k_4u_5 - k_5u_2u_6 + k_6u_4,\\
\frac{\partial u_6}{\partial t} - a_6\Delta u_6 &= -k_5u_2u_6 + k_6u_4.
\end{aligned} \tag{5.2}$$

In the special case $k_5 = k_6 = 0$, U_1, U_2, U_3, U_4, U_5 may represent hemoglobin Hb, O_2, HbO_2, CO_2 and $HbCO_2$. Hollis [6] established global existence provided that

(1) u_3 satisfies the same type of boundary conditions as either u_1 or u_2 and
(2) u_5 satisfies the same type of boundary condition as either u_1 or u_4. The results obtained in Kouachi [2] and in Kouachi *et al* [1] are not applicable. Our generalization is summarized as follows.

Proposition 3. *Solutions of (5.2) with non-negative uniformly bounded initial data and boundary conditions (1.2) exist globally.*

Proof. Condition (1.5) is satisfied for the six-component system when choosing $D_3 < D_2$ and $0 < D_5 - D_4 < D_1$. Then, corollary 1 implies that u_1, u_2, u_3, u_4, u_5 and u_6 are in $L^\infty(0, T^*; L^N(\Omega))$ for all $N \geqslant 1$. So, solutions of (5.2) exist globally. □

Example 2. We next consider a general reaction mechanism of the form:

$$\mu_1 R_1 + \mu_2 R_2 + \cdots + \mu_r R_r \underset{k_r}{\overset{k_f}{\rightleftarrows}} \nu_1 P_1 + \nu_2 P_2 + \cdots + \nu_\ell P_\ell, \tag{5.3}$$

where R_i and P_i represent reactant and product species, respectively, and μ_i, ν_i are positive constants for each i. Now, if we set $u_i = [R_i]$ and $v_i = [P_i]$ and let k_f, k_r be the (non-negative) forward and reverse reaction rates, respectively, then we may model the process by the application of the law of conservation of mass and the second law of Fick (flow) (see Kouachi [26]) with the following reaction–diffusion system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \nabla \cdot (a_i \nabla u_i) = \mu_i \left(k_r \displaystyle\prod_{j=1}^{\ell} v_j^{\nu_j} - k_f \prod_{j=1}^{r} u_j^{\mu_j} \right), & i = 1, \ldots, r \\ \dfrac{\partial v_i}{\partial t} - \nabla \cdot (a_{r+i} \nabla v_i) = \nu_i \left(k_f \displaystyle\prod_{j=1}^{r} u_j^{\mu_j} - k_r \prod_{j=1}^{\ell} v_j^{\nu_j} \right), & i = 1, \ldots, \ell, \end{cases} \tag{5.4}$$

with boundary conditions (1.2) and positive initial data in $L^\infty(\Omega)$.

In the special case when $r = 2$ and $\ell = 1$, the special case $\mu_1 = \mu_2 = \nu_1 = 1$ has been studied by Rothe (see [25, p 157]) under homogeneous Neumann boundary conditions where he showed that $T_{\max} = \infty$ if $n \leqslant 5$. Morgan [5] generalized the results of Rothe for every integer $n \geqslant 1$ and when all the components satisfy the same boundary conditions (Neumann or Dirichlet). Hollis [6] completed the work of Morgan and established global existence if u_3 satisfies the same type of boundary conditions as either u_1 or u_2. But if boundary conditions of different types are imposed on u_1 and u_2, the global existence follows regardless of the type of boundary condition that is imposed on u_3. Recently, Kouachi has proved, in [2], the global existence of solutions with boundary conditions (1.2) when $\mu_1 + \mu_2 \leqslant 1$ or $\nu_1 \leqslant 1$, and as a completion to this, we have proved the global existence of the system when $r = 2$ and $\ell = 2$, in Kouachi *et al* [1] such that $\mu_1 + \mu_2 \leqslant 1$ or $\nu_1 + \nu_2 \leqslant 1$.

By applying the obtained results on our system, we get the following proposition:

Proposition 4. *Solutions of (5.4) with non-negative uniformly bounded initial data (1.3) and nonhomogeneous boundary conditions (1.2) are positive and exist globally for every positive constant μ_i, $i = 1, \ldots, r$, and ν_i, $i = 1, \ldots, \ell$, such that* $\min\{\sum_{i=1}^{r} \mu_i, \sum_{i=1}^{\ell} \nu_i\} \leqslant 1$.

Proof. We remark that (1.4) for this system is satisfied for all positive constants μ_i, $i = 1, \ldots, r$, and ν_i, $i = 1, \ldots, \ell$, whenever

$$N = \max \left\{ \sum_{i=1}^{r} \mu_i, \sum_{i=1}^{\ell} \nu_i \right\}, \tag{5.5}$$

and condition (1.5) is trivial when $\sum_{i=1}^{r} \mu_i \leqslant 1$ by choosing $\sum_{i=r+1}^{r+\ell-1} D_i + 1 \gg \sum_{i=1}^{r} D_i$, and by applying Young's inequality to the term $\prod_{j=1}^{r} u_j^{\mu_j}$. In the case $\sum_{i=1}^{\ell} \nu_i \leqslant 1$, it is also a trivial application of Young's inequality to the term $\prod_{j=1}^{\ell} v_j^{\nu_j}$ and choosing $\sum_{i=1}^{r} D_i \gg \sum_{i=r+1}^{r+\ell-1} D_i + 1$ (see [3] for more details). Then, corollary 2 implies that all components of the solution are in $L^\infty(0, T^*; L^n(\Omega))$ for all $n \geqslant 1$, then $T_{\max} = +\infty$. □

Appendix

Proof of lemma 1. Differentiating H_{p_m} with respect to u_1 yields

$$\partial_{u_1} H_{p_m} = \sum_{p_{m-1}=1}^{p_m} \cdots \sum_{p_2=1}^{p_3} \sum_{p_1=1}^{p_2} p_1 C_{p_m}^{p_{m-1}} \cdots C_{p_3}^{p_2} C_{p_2}^{p_1} \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2}$$
$$\times u_1^{p_1 - 1} u_2^{p_2 - p_1} u_3^{p_3 - p_2} \cdots u_m^{p_m - p_{m-1}}.$$

Using the fact that

$$p_i C_{p_{i+1}}^{p_i} = p_{i+1} C_{p_{i+1}-1}^{p_i-1} \tag{A.1}$$

for all $i = 1, \ldots, m-1$ we get

$$\partial_{u_1} H_{p_m} = p_m \sum_{p_{m-1}=1}^{p_m} \cdots \sum_{p_2=1}^{p_3} \sum_{p_1=1}^{p_2} C_{p_m-1}^{p_{m-1}-1} \cdots C_{p_3-1}^{p_2-1} C_{p_2-1}^{p_1-1} \theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2}$$
$$\times u_1^{p_1-1} u_2^{p_2-p_1} u_3^{p_3-p_2} \cdots u_m^{p_m-p_{m-1}},$$

while changing in the sums the indices $p_i - 1$ by p_i for all $i = 1, \ldots, m-1$, we deduce (4.1). For formulae (4.2) and (4.3) and differentiating H_{p_m} with respect to u_i, $i = 2, \ldots, m$, gives

$$\partial_{u_i} H_n = \sum_{p_{m-1}=1}^{p_m} \cdots \sum_{p_i=1}^{p_{i+1}} \cdots \sum_{p_1=0}^{p_2} (p_i - p_{i-1}) C_{p_m}^{p_{m-1}} \cdots C_{p_i}^{p_{i-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2}$$
$$\times u_1^{p_1} u_2^{p_2-p_1} \cdots u_i^{p_i-p_{i-1}-1} \cdots u_m^{p_m-p_{m-1}}.$$

Taking account of

$$C_{p_i}^{p_{i-1}} = C_{p_i}^{p_i-p_{i-1}}, \quad p_{i-1} = 0, \ldots, p_i - 1 \quad \text{and} \quad p_i = 1, \ldots, p_{i+1} \tag{A.2}$$

using (A.1) and changing the index $p_i - 1$ by p_i for all $i = 1, \ldots, m-1$, we get (4.2) and (4.3). □

Proof of lemma 2. Differentiating $\partial_{u_1} H_{p_m}$, given by formula (4.1), with respect to u_1 yields

$$\partial_{u_1^2} H_{p_m} = p_m \sum_{p_{m-1}=1}^{p_m-1} \cdots \sum_{p_1=1}^{p_2} p_1 C_{p_m-1}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{(p_1+1)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+1)^2}$$
$$\times u_1^{p_1-1} u_2^{p_2-p_1} \cdots u_m^{(p_m-1)-p_{m-1}}.$$

Using (A.1) and while changing in the sums the indices $p_i - 1$ by p_i for all $i = 1, \ldots, m-1$, we get (4.4).

For all $i = 2, \ldots, m-1$,

$$\partial_{u_i^2} H_{p_m} = p_m \sum_{p_{m-1}=1}^{p_m-1} \cdots \sum_{p_i=1}^{p_{i+1}} \cdots \sum_{p_1=0}^{p_2} (p_i - p_{i-1}) C_{p_m-1}^{p_{m-1}} \cdots C_{p_i}^{p_{i-1}} \cdots C_{p_2}^{p_1}$$
$$\times \theta_1^{p_1^2} \cdots \theta_{i-1}^{p_{(i-1)}^2} \theta_i^{(p_i+1)2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+1)^2} u_1^{p_1} u_2^{p_2-p_1} \cdots u_3^{p_i-p_{i-1}-1} \cdots u_m^{(p_m-1)-p_{m-1}}.$$

Applying (A.2) and then (A.1), we get (4.5).

Differentiating $\partial_{u_i} H_{p_m}$ given by formula (4.2) with respect to u_j yields

$$\partial_{u_j u_i} H_{p_m} = \partial_{u_j} \left(\partial_{u_i} H_n \right)$$
$$= p_m \sum_{p_{m-1}=1}^{p_m-1} \cdots \sum_{p_i=1}^{p_{i+1}} \cdots \sum_{p_j=1}^{p_{j+1}} \cdots \sum_{p_1=0}^{p_2} (p_j - p_{j-1}) C_{p_m-1}^{p_{m-1}} \cdots C_{p_i}^{p_{i-1}} \cdots C_{pj}^{p_{j-1}} \cdots C_{p_2}^{p_1}$$
$$\times \theta_1^{p_1^2} \cdots \theta_j^{p_j^2} \cdots \theta_i^{(p_i+1)^2} \cdots \theta_{(m-1)}^{(p_{(m-1)}+1)^2}$$
$$\times u_1^{p_1} u_2^{p_2-p_1} \cdots u_j^{p_j-p_{j-1}-1} \cdots u_i^{p_i-p_{i-1}} \cdots u_m^{(p_m-1)-p_{m-1}},$$

for all $i = 2, \ldots, m-1$.

Applying successively (A.2), (A.1) and (A.2) a second time, we deduce (4.6).

Finally, we get (4.7) by differentiating $\partial_{u_m} H_{p_m}$ with respect to u_m and applying successively (A.2), (A.1) and (A.2) a second time. □

Proof of lemma 3. We prove this lemma inductively.

For $m = 2$, we have $K_2^2 = \det 2$.

For $m = 3$, by using the well-known Dodgson condensation which has been modified [13] on the symmetric 3-square matrix,

$$\det 1 \det 3 = \det 2 \det_{1\leqslant i,j\leqslant 3}\left[(a_{i,j})_{\substack{i\neq 2\\ j\neq 2}}\right] - \left[\det_{1\leqslant i,j\leqslant 3}\left[(a_{i,j})_{\substack{i\neq 3\\ j\neq 2}}\right]\right]^2, \tag{A.3}$$

but

$$\det 2 = K_2^2, \qquad \det_{1\leqslant i,j\leqslant 3}\left[(a_{i,j})_{\substack{i\neq 2\\ j\neq 2}}\right] = a_{11}a_{33} - (a_{13})^2 = K_3^2$$

and

$$\det_{1\leqslant i,j\leqslant 3}\left[(a_{i,j})_{\substack{i\neq 3\\ j\neq 2}}\right] = a_{11}a_{23} - a_{12}a_{13} = H_3^2.$$

Formula (A.3) will become

$$\det 1 \det 3 = K_2^2 \cdot K_3^2 - \left[H_3^2\right]^2.$$

So by using formula (4.8*a*), formula (4.8) will be correct for $m = 3$.

When $m \geqslant 4$, we suppose formula (4.8) is correct for $m-1, m-2, m-3, \ldots, 4$ and we prove it for m.

It is sufficient to prove

$$K_m^{m-1} = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m-1\\ j\neq m-1}}\right) \cdot \prod_{k=1}^{k=m-3} (\det k)^{2^{(m-K-3)}}. \tag{A.4}$$

By putting $l = m-1$ in formula (4.8*b*), we get

$$H_m^{m-1} = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m\\ j\neq m-1}}\right) \cdot \prod_{k=1}^{k=m-3} (\det k)^{2^{(m-K-3)}}. \tag{A.5}$$

From the inductive proof, we have

$$K_{(m-1)}^{(m-1)} = \det(m-1) \cdot \prod_{k=1}^{k=m-3} (\det k)^{2^{(m-K-3)}}. \tag{A.6}$$

By putting $l = m$ in formula (4.8*a*), we get

$$K_m^m = K_{m-1}^{m-1} \cdot K_m^{m-1} - \left(H_m^{m-1}\right)^2, \tag{A.7}$$

by replacing (A.4), (A.5) and (A.6) in (A.7), we get

$$K_m^m = \prod_{k=1}^{k=m-3} (\det k)^{2^{(m-K-2)}} \cdot \det(m-2) \cdot \det m$$

$$= \det m \cdot \prod_{k=1}^{k=m-2} (\det k)^{2^{(m-K-2)}}$$

and thus formula (4.8) is correct for m.

Now, we prove formula (A.4); we may generalize the formula as follows:

$$K_m^l = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m-1,\ldots,l\\ j\neq m-1,\ldots,l}}\right)\cdot\prod_{k=1}^{k=l-2}(\det k)^{2^{((l-2)-K)}},\qquad l=3,\ldots,m-1. \tag{A.8}$$

Also, we prove formula (A.8) inductively. It is a secondary unductively proof included in the principle inductive proof.

For $l = 2$, it is evident.

For $l = 3$, formula (4.8a) will become

$$K_m^3 = K_2^2\cdot K_m^2 - \left(H_m^2\right)^2.$$

It is evident to make sure that the following are correct:

$$K_2^2 = \det 2,$$

$$K_m^2 = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m-1,\ldots,2\\ j\neq m-1,\ldots,2}}\right),$$

$$H_m^2 = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m-1,\ldots,2\\ j\neq m,\ldots,3}}\right).$$

By using the well-known Dodgson condensation which has been modified, we get formula (A.8) for $l = 3$.

When $l \geqslant 4$, we suppose formula (A.8) is correct for $l-1$ and we prove it for l. Formula (4.8a) will become

$$K_m^l = K_{l-1}^{l-1}\cdot K_m^{l-1} - \left(H_m^{l-1}\right)^2. \tag{A.9}$$

According to the principle of inductive proposition, we have

$$K_{(l-1)}^{(l-1)} = \det(l-1)\cdot\prod_{k=1}^{k=l-3}(\det k)^{2^{(l-K-3)}}; \tag{A.10}$$

according to the secondary inductive proposition, we have

$$K_m^{l-1} = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m-1,\ldots,l-1\\ j\neq m-1,\ldots,l-1}}\right)\cdot\prod_{k=1}^{k=(l-3)}(\det k)^{2^{((l-3)-K)}}; \tag{A.11}$$

according to formula (4.8b), we have

$$H_m^{l-1} = \det_{1\leqslant i,j\leqslant m}\left((a_{i,j})_{\substack{i\neq m,\ldots,l\\ j\neq m-1,\ldots,l-1}}\right)\cdot\prod_{k=1}^{k=l-3}(\det k)^{2^{((l-3)-K)}} \tag{A.12}$$

by replacing (A.10), (A.11) and (A.12) in (A.9) and by using the well-known Dodgson condensation which has been modified, we get formula (A.8) for l. This completes the secondary inductive proof and thus also the proof of the main theorem. □

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